Relativistic addition of perpendicular velocity components from the constancy of the speed of light

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Abstract

Mermin [Am. J. Phys. **51**, 1130–1131 (1983)] derived the relativistic addition of the parallel components of velocity using the constancy of the speed of light. In this note, the derivation is extended to the perpendicular components of velocity.

Mermin[1] gave a succinct elementary derivation of the relativistic addition the parallel component of the velocity using the constancy of the speed of light, without resorting to Lorentz transformations. Here, I show that his derivation can be extended to the addition of the perpendicular components of the velocity.

As in Ref. 1, we consider a train moving on a long straight track at constant speed v. At a certain instant, a photon, which has speed c, and a massive particle, which has speed less than c, are both projected from one side the train, say the right side, in the direction perpendicular (according to someone standing on the train) to the direction of motion of the train. When the photon reaches the left side of the train, it is immediately reflected back. On its way back to the right side, it encounters the massive particle at some point.

Since the place on the train where the photon and the massive particle meet is frame-independent, the ratio r of the perpendicular components of the velocity of the massive particle, $w_\perp$, and that of the photon must be an invariant. To a person standing on the ground, the parallel component of the photon's velocity is v, the speed of the train. Since the speed of the photon is constant at c in any frame, the perpendicular component of the velocity according to the person standing on the ground is, by Pythagoras' theorem, $\sqrt{c^2 - v^2}$. Therefore, the invariant ratio is

$$r = \frac{w_\perp}{\sqrt{c^2 - v^2}} \,. \tag{1}$$

Consider two cases for the speed of the train, v and v'. The invariance of r implies

$$\begin{aligned} \frac{w_\perp}{\sqrt{c^2 - v^2}} &= \frac{w'_\perp}{\sqrt{c^2 - v'^2}} \\ \Rightarrow w'_\perp &= w_\perp \sqrt{\frac{1 - v'^2/c^2}{1 - v^2/c^2}}. \end{aligned} \tag{2}$$

All that is left to do is to express v' in terms of the velocity difference V between v' and v using the parallel velocity addition rule[1]

$$v' = \frac{v + V}{1 + vV/c^2}. \tag{3}$$

Substituting this into Eq. (2) gives the desired result

$$w'_\perp = w_\perp \frac{\sqrt{1 - V^2/c^2}}{1 + vV/c^2}. \tag{4}$$

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[1] N. David Mermin, "Relativistic addition of velocities directly from the constancy of the velocity of light," Am. J. Phys. **51**, 1130–1131 (1983).